The η' and the topological charge density [1]

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Abstract

We compare two different frameworks which have been proposed to include the η' in chiral perturbation theory. The equivalence of these two approaches is shown both for the purely mesonic case and in the presence of the ground state baryon octet. The relation between the different sets of parameters in both Lagrangians is clarified.

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1 Introduction

The QCD Lagrangian with massless quarks exhibits an $SU(3)_R \times SU(3)_L$ chiral symmetry which is broken down spontaneously to $SU(3)_V$, giving rise to a Goldstone boson octet of pseudoscalar mesons which become massless in the chiral limit of zero quark masses. On the other hand, the axial $U(1)$ symmetry of the QCD Lagrangian is broken by the anomaly so that the mass of the corresponding pseudoscalar singlet does not vanish in the chiral limit. The lightest candidate would be the η' with a mass of 958 MeV which is considerably heavier than the octet states. In conventional chiral perturbation theory the η' is not included explicitly, although it does show up in the form of a contribution to a coupling coefficient of the Lagrangian, a so-called low-energy constant (LEC).

However, experiment suggests that the physical states — η and η' — are mixtures of octet and singlet components. In order to include this effect in chiral perturbation theory one should treat the η' as a dynamical field variable instead of integrating it out from the effective theory. This approach is also motivated by large N_c considerations. In this limit the axial anomaly is supressed by powers of $1/N_c$ and gives rise to a ninth Goldstone boson, the η'. The inclusion of the η' in chiral perturbation theory has been the subject of previous work, see e.g. [1, 2, 3]. But while the authors of [1, 2] work with a $U(3)_R \times U(3)_L$ invariant Lagrangian, gluonic terms have been included explicitly in the effective theory in [3]. The equivalence of both approaches is rather evident to lowest order both in the chiral and the $1/N_c$ expansion, e.g., the Lagrangian in Eq. (2.22) of Di Vecchia's and Veneziano's work [3] coincides with the corresponding part of Eq. (2) in Leutwyler's presentation [2]. But so far no systematic comparison between both schemes has been made to prove the equivalence at higher orders. The purpose of the present work is to fill this gap.

In the next two sections we will compare the mesonic Lagrangians in both approaches and, furthermore, generalize the approach of [3] to higher orders in the gluonic terms. Having shown the equivalence of both frameworks in the mesonic sector, we proceed by including the ground state baryon octet in Section 4. The inclusion of the η' in baryon chiral perturbation theory has been the subject of recent work [4, 5] and again the two different approaches have been used without clarifying the connection between both schemes. It is therefore desirable to show the equivalence also in the baryonic case. We conclude with a short summary.

2 The $U(1)_A$ invariant effective Lagrangian

In this section we will briefly outline the method of extending the $SU(3)_R \times SU(3)_L$ symmetry of the effective Lagrangian in conventional chiral perturbation theory to $U(3)_R \times U(3)_L$ in a more generalized framework including the η', see

e.g. [1, 2]. Within this approach the topological charge operator coupled to an external field is added to the QCD Lagrangian

$$\mathcal{L} = \mathcal{L}_{QCD} - \frac{g^2}{16\pi^2}\theta(x)\mathrm{tr}_c(G_{\mu\nu}\tilde{G}^{\mu\nu}) \tag{1}$$

with $\tilde{G}_{\mu\nu} = \epsilon_{\mu\nu\alpha\beta}G^{\alpha\beta}$ and tr_c is the trace over the color indices. Under $U(1)_R \times U(1)_L$ the axial $U(1)$ anomaly adds a term $-(g^2/16\pi^2)2N_f\,\alpha\,\mathrm{tr}_c(G_{\mu\nu}\tilde{G}^{\mu\nu})$ to the QCD Lagrangian, with N_f being the number of different quark flavors and α the angle of the global axial $U(1)$ rotation. The vacuum angle $\theta(x)$ is in this context treated as an external field that transforms under an axial $U(1)$ rotation as

$$\theta(x) \rightarrow \theta'(x) = \theta(x) - 2N_f\alpha. \tag{2}$$

Then the term generated by the anomaly in the fermion determinant is compensated by the shift in the θ source and the Lagrangian from Eq. (1) remains invariant under axial $U(1)$ transformations. The symmetry group $SU(3)_R \times SU(3)_L$ of the Lagrangian $\mathcal{L}_{QCD}$ is extended to $U(3)_R \times U(3)_L$ for $\mathcal{L}$. [3] This property remains at the level of an effective theory and the additional source θ also shows up in the effective Lagrangian. Let us consider the purely mesonic effective theory first. The lowest lying pseudoscalar meson nonet is summarized in a matrix valued field $U(x)$

$$U(\phi,\eta_0) = u^2(\phi,\eta_0) = \exp\{2i\phi/F_\pi + i\sqrt{\frac{2}{3}}\eta_0/F_0\}, \tag{3}$$

where $F_\pi \simeq 92.4$ MeV is the pion decay constant and the singlet η_0 couples to the singlet axial current with strength F_0. The unimodular part of the field $U(x)$ contains the degrees of freedom of the Goldstone boson octet ϕ

$$\phi = \frac{1}{\sqrt{2}}\begin{pmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta_8 & K^0 \\ K^- & \bar{K^0} & -\frac{2}{\sqrt{6}}\eta_8 \end{pmatrix} \quad , \tag{4}$$

while the phase $\det U(x) = e^{i\sqrt{6}\eta_0/F_0}$ describes the η_0 . The symmetry $U(3)_R \times U(3)_L$ does not have a dimension-nine irreducible representation and consequently does not exhibit a nonet symmetry. We have therefore used the different notation F_0 for the decay constant of the singlet field. The effective Lagrangian is formed with the fields $U(x)$, derivatives thereof and also includes both the quark mass matrix $\mathcal{M}$ and the vacuum angle θ: $\mathcal{L}_{\mathrm{eff}}(U, \partial U, \ldots, \mathcal{M}, \theta)$. Under $U(3)_R \times U(3)_L$ the fields transform as follows

$$U' = RUL^\dagger \quad , \qquad \mathcal{M}' = R\mathcal{M}L^\dagger \quad , \qquad \theta'(x) = \theta(x) - 2N_f\alpha \tag{5}$$

[3] Note that the Lagrangian actually changes by a total derivative which gives rise to the Wess-Zumino term. We will disregard this contribution, since it is irrelevant for proving the equivalence of both schemes discussed in this presentation.

with $R \in U(3)_R$, $L \in U(3)_L$, but the Lagrangian remains invariant. The phase of the determinant $\det U(x) = e^{i\sqrt{6}\eta_0/F_0}$ transforms under axial $U(1)$ as $\sqrt{6}\eta_0'/F_0 = \sqrt{6}\eta_0/F_0 + 2N_f\alpha$ so that the combination $\sqrt{6}\eta_0/F_0 + \theta$ remains invariant. It is more convenient to replace the variable θ by this invariant combination, $\mathcal{L}_{\rm eff} = \mathcal{L}_{\rm eff}(U, \partial U, \ldots, \mathcal{M}, \sqrt{6}\eta_0/F_0 + \theta)$. One can now construct the effective Lagrangian in these fields that respects the symmetries of the underlying theory. In particular, the Lagrangian is invariant under $U(3)_R \times U(3)_L$ rotations of U and $\mathcal{M}$ at a fixed value of the last argument. The most general Lagrangian up to and including terms with two derivatives and one factor of $\mathcal{M}$ reads

$$\begin{aligned}\mathcal{L}_\phi &= -V_0 + V_1\langle\nabla_\mu U^\dagger\nabla^\mu U\rangle + V_2\langle\chi_+\rangle + iV_3\langle\chi_-\rangle \\ &\quad +V_4\langle U^\dagger\nabla_\mu U\rangle\langle U^\dagger\nabla^\mu U\rangle + iV_5\langle U^\dagger\nabla_\mu U\rangle\nabla^\mu\theta + V_6\nabla_\mu\theta\nabla^\mu\theta. \qquad (6)\end{aligned}$$

The expression $\langle\ldots\rangle$ denotes the trace in flavor space and the quark mass matrix $\mathcal{M} = \mathrm{diag}(m_u, m_d, m_s)$ enters in the combinations

$$\chi_\pm = 2B_0(u\mathcal{M}u \pm u^\dagger\mathcal{M}u^\dagger) \qquad (7)$$

with $B_0 = -\langle 0|\bar{q}q|0\rangle/F_\pi^2$ the order parameter of the spontaneous symmetry violation. The covariant derivatives are defined by

$$\begin{aligned}\nabla_\mu U &= \partial_\mu U - i(v_\mu + a_\mu)U + iU(v_\mu - a_\mu) \\ \nabla_\mu\theta &= \partial_\mu\theta + 2\langle a_\mu\rangle. \qquad (8)\end{aligned}$$

The external fields $v_\mu(x), a_\mu(x)$ represent hermitian 3×3 matrices in flavor space. Note that the term of the type $i\langle U^\dagger\nabla_\mu U\rangle\nabla^\mu\theta$ can be transformed away [2], but for our purposes it is more convenient to keep this term explicitly. Once the equivalence of both approaches is shown, one is free to eliminate such a term. The coefficients V_i are functions of the variable $\sqrt{6}\eta_0/F_0 + \theta$, $V_i(\sqrt{6}\eta_0/F_0 + \theta)$, and can be expanded in terms of this variable. At a given order of derivatives of the meson fields U and insertions of the quark mass matrix $\mathcal{M}$ one obtains an infinite string of increasing powers of the singlet field η_0 with couplings which are not fixed by chiral symmetry. Parity conservation implies that the V_i are all even functions of $\sqrt{6}\eta_0/F_0+\theta$ except V_3, which is odd, and $V_1(0) = V_2(0) = F_\pi^2/4$ gives the correct normalizaton for the quadratic terms of the Goldstone boson octet.

3 The topological charge density within an effective Lagrangian

In the literature, another approach of incorporating the axial $U(1)$ anomaly in an effective Lagrangian can be found [3]. But so far no attempt has been made to compare this scheme with the approach presented in the last section. In

this section we will set up an effective Lagrangian following the ideas of [3] and compare it with the Lagrangian from Eq. (6). The starting point is the effective Lagrangian

$$\mathcal{L}_\phi = \frac{F_\pi^2}{4}\langle \nabla_\mu U^\dagger \nabla^\mu U\rangle + \frac{F_\pi^2}{4}\langle \chi_+\rangle + a\langle U^\dagger \nabla_\mu U\rangle\langle U^\dagger \nabla^\mu U\rangle \tag{9}$$

which reduces to conventional $SU(3)_R \times SU(3)_L$ chiral perturbation theory if the singlet field η_0 is neglected. A new low-energy constant a enters the calculation which for our purposes here will be left undetermined. Next, one introduces gluonic terms in order to reproduce the anomaly in the divergence of the axial-vector current

$$\partial_\mu J_5^\mu = 2i\sum_f m_f \bar{q}_f \gamma_5 q_f + \frac{g^2}{16\pi^2} 2N_f \mathrm{tr}_c(G_{\mu\nu}\tilde{G}^{\mu\nu}) \tag{10}$$

by defining $Q(x) \equiv (g^2/16\pi^2)\mathrm{tr}_c(G_{\mu\nu}\tilde{G}^{\mu\nu})$. The correct transformation under axial $U(1)$ is achieved by adding the term

$$\delta\mathcal{L} = \frac{i}{2}Q\langle \log U - \log U^\dagger\rangle \tag{11}$$

to the effective Lagrangian, where it is assumed that the topological charge density $Q(x)$ remains invariant under $U(1)_A$ transformations. The most general effective Lagrangian in this framework up to and including terms with two derivatives, one factor of $\mathcal{M}$ and quadratic terms in Q respecting the symmetries of the underlying theory reads

$$\begin{aligned}
\mathcal{L}_\phi \;=\; & \Big(\frac{F_\pi^2}{4} + v_1 Q^2\Big)\langle \nabla_\mu U^\dagger \nabla^\mu U\rangle + \Big(\frac{F_\pi^2}{4} + v_2 Q^2\Big)\langle \chi_+\rangle + \kappa Q \\
& + \frac{i}{2}Q\langle \log U - \log U^\dagger\rangle + \tau Q^2 + i v_3 Q\langle \chi_-\rangle + v_6 \partial_\mu Q \partial^\mu Q \\
& + \Big(a + v_4 Q^2\Big)\langle U^\dagger \nabla_\mu U\rangle\langle U^\dagger \nabla^\mu U\rangle + i v_5 \langle U^\dagger \nabla_\mu U\rangle \partial^\mu Q
\end{aligned} \tag{12}$$

where an irrelevant constant has been omitted. From matching to QCD we know that the parity-violating piece κQ of the effective Lagrangian equals

$$\delta\mathcal{L} = -\theta\frac{g^2}{16\pi^2}\mathrm{tr}_c(G_{\mu\nu}\tilde{G}^{\mu\nu}) = -\theta Q. \tag{13}$$

We will therefore set $\kappa = -\theta$ in the following. Usually authors have neglected some of these terms using only the Lagrangian

$$\begin{aligned}
\mathcal{L}'_\phi \;=\; & \frac{F_\pi^2}{4}\langle \nabla_\mu U^\dagger \nabla^\mu U\rangle + \frac{F_\pi^2}{4}\langle \chi_+\rangle + a\langle U^\dagger \nabla_\mu U\rangle\langle U^\dagger \nabla^\mu U\rangle \\
& + \frac{i}{2}Q\langle \log U - \log U^\dagger\rangle - \theta Q + \tau Q^2
\end{aligned} \tag{14}$$

in which Q decouples from the Goldstone boson octet ϕ. This is motivated by the fact that the topological charge density Q behaves in the large N_c limit as $Q \propto g^2 \propto 1/N_c$ and higher orders of Q are suppressed by powers of $1/N_c$. In order to prove the equivalence of this approach to that of the last section, we prefer to work with the Lagrangian in Eq. (12). The generalization of the proof to higher orders, both in the derivative expansion and in Q, is straightforward and will be discussed later. We will therefore restrict ourselves to this Lagrangian in the beginning. The gluonic term Q is treated as a background field and is integrated out from the Lagrangian via its equation of motion

$$\partial_\mu \frac{\delta \mathcal{L}}{\delta \partial_\mu Q} - \frac{\delta \mathcal{L}}{\delta Q} = 0. \tag{15}$$

To lowest order in the derivatives and the quark masses the equation of motion for Q reads

$$\begin{aligned} Q &= \frac{1}{2\tau}\Big(\theta - \frac{i}{2}\langle \log U - \log U^\dagger \rangle\Big) \\ &= \frac{1}{2\tau}\Big(\theta + \sqrt{6}\eta_0/F_0\Big) \equiv \frac{1}{2\tau} Q_0. \end{aligned} \tag{16}$$

Under axial $U(1)$ tansformations the η_0 field transforms as $\sqrt{6}\eta_0/F_0 \to \sqrt{6}\eta_0/F_0 + 2N_f\alpha$, where α is the angle of the axial $U(1)$ rotation. For Q to remain invariant, θ has to compensate for the change in η_0, cf. Eq. (2),

$$\theta \to \theta - 2N_f\alpha. \tag{17}$$

It is therefore more convenient to consider θ as an external field $\theta(x)$ which has under $U(1)_A$ the transformation property given in Eq. (17) rather than to treat it as a constant (see the work by Di Vecchia and Veneziano [3] for the latter case). This leads to an effective Lagrangian which remains invariant also under $U(1)_A$ rotations in agreement with the first approach. Otherwise, Q would not be $U(1)_A$ invariant in contradiction to the assumption. Reinserting the solution $\frac{1}{2\tau}Q_0$ of the equation of motion for Q into the Lagrangian in Eq. (12) one obtains

$$\begin{aligned} \mathcal{L}_\phi &= \Big(\frac{F_\pi^2}{4} + \frac{v_1}{4\tau^2}Q_0^2\Big)\langle \nabla_\mu U^\dagger \nabla^\mu U\rangle + \Big(\frac{F_\pi^2}{4} + \frac{v_2}{4\tau^2}Q_0^2\Big)\langle \chi_+\rangle - \frac{1}{4\tau}Q_0^2 \\ &\quad + i\frac{v_3}{2\tau}Q_0\langle\chi_-\rangle + \Big(a + \frac{v_5}{2\tau} - \frac{v_6}{4\tau^2} + \frac{v_4}{4\tau^2}Q_0^2\Big)\langle U^\dagger \nabla_\mu U\rangle\langle U^\dagger \nabla^\mu U\rangle \\ &\quad + i\Big(\frac{v_5}{2\tau} - \frac{v_6}{2\tau^2}\Big)\langle U^\dagger \nabla_\mu U\rangle \nabla^\mu\theta + \frac{v_6}{4\tau^2}\nabla_\mu\theta\nabla^\mu\theta. \end{aligned} \tag{18}$$

This Lagrangian is in complete agreement with the one in Eq. (6), once one expands the functions V_i in powers of $\sqrt{6}\eta_0/F_0 + \theta = Q_0$ and keeps only the first terms in the expansions. There is a one-to-one correspondence between the low-energy constants in both schemes to the order we are working. This

equivalence is maintained at higher orders both in the derivative expansion and in the background field Q.

Firstly, we will examine the latter case by adding a piece $\delta\mathcal{L} = \lambda Q^4$ to the Lagrangian. Other terms with higher powers of Q can be included in the Lagrangian as well, but they do not alter the following considerations. In order to keep the presentation lucid, we restrict ourselves to this simple extension. The modified equation of motion for Q reads to leading order in the derivatives and quark masses

$$-Q_0 + 2\tau Q + 4\lambda Q^3 = 0. \tag{19}$$

Although this equation can still be solved analytically, we prefer to solve it perturbatively, since this method can be generalized to arbitrary high powers in Q. The $1/N_c$ expansion provides the perturbative framework for solving the equation of motion if higher powers of Q are included. To next-to-leading order in $1/N_c$ one can write

$$Q = \frac{1}{2\tau} Q_0 + \delta Q \tag{20}$$

and Eq. (19) leads then to

$$\delta Q = -\frac{\lambda}{4\tau^4} Q_0^3 \tag{21}$$

modulo higher corrections in $1/N_c$, i.e. higher orders of Q_0. Reinserting the solution for Q into the effective Lagrangian one obtains a similar Lagrangian as in Eq. (18), but with higher orders in $Q_0 = \sqrt{6}\eta_0/F_0 + \theta$ which for the sake of brevity are not shown here. Therefore, going up to higher powers of Q is similar to expanding the functions V_i to higher orders in $\sqrt{6}\eta_0/F_0 + \theta$. Having examined the impact of higher orders of Q in the effective Lagrangian, we will restrict ourselves to the Lagrangian with factors of Q and Q^2 given in Eq. (12).

So far we have eliminated the field Q via its equation of motion at lowest order in the derivatives and quark masses. We will now proceed by including a term of higher chiral order into the equation of motion. In order to keep the arguments as simple as possible we restrict ourselves to the term $i\langle U^\dagger \nabla_\mu U\rangle \partial^\mu Q$. The inclusion of further terms such as $iQ\langle \chi_-\rangle$ is straightforward and can be treated in a similar way. The equation of motion is then derived from the Lagrangian

$$\delta\mathcal{L} = \frac{i}{2} Q \langle \log U - \log U^\dagger \rangle - \theta Q + \tau Q^2 + i v_5 \langle U^\dagger \nabla_\mu U\rangle \partial^\mu Q \tag{22}$$

and reads

$$i v_5 \partial_\mu \langle U^\dagger \nabla^\mu U \rangle = -Q_0 + 2\tau Q. \tag{23}$$

We can decompose the solution for Q into the piece at lowest chiral order $\frac{1}{2\tau}Q_0$ and a small perturbation ΔQ

$$Q = \frac{1}{2\tau} Q_0 + \Delta Q \tag{24}$$

so that

$$\Delta Q = \frac{i}{2\tau} v_5 \partial_\mu \langle U^\dagger \nabla^\mu U \rangle. \tag{25}$$

Inserting Q into the Lagrangian in Eq. (12) and neglecting terms of higher chiral orders, the only additional terms linear in ΔQ read

$$\delta\mathcal{L} = -\Big(\sqrt{6}\eta_0/F_0 + \theta\Big)\Delta Q + \tau 2\frac{Q_0}{2\tau}\Delta Q = 0. \tag{26}$$

Therefore, taking only the term $iv_5\langle U^\dagger \nabla_\mu U\rangle \partial^\mu Q$ into account and working to second order in the derivative expansion, the additional terms in the Lagrangian happen to cancel. But in general the procedure of eliminating Q via its equation of motion perturbatively in the derivative or quark mass expansion will produce terms of higher chiral orders and will lead to the renormalization of the pertinent couplings of such terms. This concludes the proof of the equivalence of the Lagrangian which explicitly includes the topological charge density with the one given in the last section up to any order both in the derivative expansion and in Q. At this point, we would like to stress that in order to prove the equivalence, it is essential that Q is eliminated via its classical equation of motion. Using the equation of motion with quantum corrections for Q would destroy the equivalence, since this would lead, e.g., to nonanalytic expressions in the quark masses which cannot be absorbed by a Lagrangian that is a polynomial in the quark mass matrix $\mathcal{M}$. Such nonanalytic terms are absent in the approach of [1, 2]. Furthermore, the quantum corrections of the equation of motion for Q are in general divergent and have to be regularized. This leads to scale dependent contributions which must be compensated by a suitable redefinition of the pertinent coupling constants.

4 Inclusion of baryons

After having ensured ourselves that the approaches discussed above are equivalent in the purely mesonic sector, we can now proceed by including the ground state baryon octet in the effective theory. To this end, it is convenient to summarize the meson fields in an object of axial-vector type with one derivative

$$u_\mu = iu^\dagger \nabla_\mu U u^\dagger. \tag{27}$$

The matrix u_μ transforms under $U(3)_R \times U(3)_L$ as a matter field,

$$u_\mu \rightarrow u'_\mu = K u_\mu K^\dagger \tag{28}$$

with $K(U, R, L)$ the compensator field representing an element of the conserved subgroup $U(3)_V$. In the context of the first scheme the baryonic Lagrangian up

to linear order in the derivative expansion has already been given in [4] and reads

$$\begin{aligned}\mathcal{L}_{\phi B} &= iW_1\langle[D^\mu,\bar{B}]\gamma_\mu B\rangle - iW_1^*\langle\bar{B}\gamma_\mu[D^\mu,B]\rangle + W_2\langle\bar{B}B\rangle\\ &\quad +W_3\langle\bar{B}\gamma_\mu\gamma_5\{u^\mu,B\}\rangle + W_4\langle\bar{B}\gamma_\mu\gamma_5[u^\mu,B]\rangle + W_5\langle\bar{B}\gamma_\mu\gamma_5 B\rangle\langle u^\mu\rangle\\ &\quad +W_6\langle\bar{B}\gamma_\mu\gamma_5 B\rangle\nabla^\mu\theta + iW_7\langle\bar{B}\gamma_5 B\rangle \end{aligned} \tag{29}$$

with D_μ being the covariant derivative of the baryon fields and the baryon octet B is given by the matrix

$$B = \begin{pmatrix} \frac{1}{\sqrt{2}}\Sigma^0 + \frac{1}{\sqrt{6}}\Lambda & \Sigma^+ & p \\ \Sigma^- & -\frac{1}{\sqrt{2}}\Sigma^0 + \frac{1}{\sqrt{6}}\Lambda & n \\ \Xi^- & \Xi^0 & -\frac{2}{\sqrt{6}}\Lambda \end{pmatrix} \tag{30}$$

which transforms as a matter field

$$B \to B' = KBK^\dagger. \tag{31}$$

The W_i are functions of the combination $\sqrt{6}\eta_0/F_0 + \theta$. From parity it follows that they are even in this variable except W_7 which is odd.

If one prefers to include the background field Q explicitly, see e.g. [5][4], the baryonic Lagrangian reads up to quadratic terms in Q

$$\begin{aligned}\mathcal{L}_{\phi B} &= i\Big(-\frac{1}{2}+\alpha Q^2\Big)\langle[D^\mu,\bar{B}]\gamma_\mu B\rangle - i\Big(-\frac{1}{2}+\alpha^* Q^2\Big)\langle\bar{B}\gamma_\mu[D^\mu,B]\rangle\\ &\quad +\Big(-\overset{\circ}{M}+\beta Q^2\Big)\langle\bar{B}B\rangle + \Big(-\frac{1}{2}D+\gamma Q^2\Big)\langle\bar{B}\gamma_\mu\gamma_5\{u^\mu,B\}\rangle\\ &\quad +\Big(-\frac{1}{2}F+\delta Q^2\Big)\langle\bar{B}\gamma_\mu\gamma_5[u^\mu,B]\rangle + \Big(-\frac{1}{2}\Lambda+\epsilon Q^2\Big)\langle\bar{B}\gamma_\mu\gamma_5 B\rangle\langle u^\mu\rangle\\ &\quad +i\kappa Q\langle\bar{B}\gamma_5 B\rangle + \lambda\langle\bar{B}\gamma_\mu\gamma_5 B\rangle\partial^\mu Q. \end{aligned} \tag{32}$$

Taking Q from the equation of motion at lowest order as given in Eq. (16) one obtains

$$\begin{aligned}\mathcal{L}_{\phi B} &= i\Big(-\frac{1}{2}+\frac{\alpha}{4\tau^2}Q_0^2\Big)\langle[D^\mu,\bar{B}]\gamma_\mu B\rangle - i\Big(-\frac{1}{2}+\frac{\alpha^*}{4\tau^2}Q_0^2\Big)\langle\bar{B}\gamma_\mu[D^\mu,B]\rangle\\ &\quad +\Big(-\overset{\circ}{M}+\frac{\beta}{4\tau^2}Q_0^2\Big)\langle\bar{B}B\rangle + \Big(-\frac{1}{2}D+\frac{\gamma}{4\tau^2}Q_0^2\Big)\langle\bar{B}\gamma_\mu\gamma_5\{u^\mu,B\}\rangle\\ &\quad +\Big(-\frac{1}{2}F+\frac{\delta}{4\tau^2}Q_0^2\Big)\langle\bar{B}\gamma_\mu\gamma_5[u^\mu,B]\rangle + i\frac{\kappa}{2\tau}Q_0\langle\bar{B}\gamma_5 B\rangle\\ &\quad +\Big(-\frac{1}{2}\Lambda-\frac{\lambda}{2\tau}+\frac{\epsilon}{4\tau^2}Q_0^2\Big)\langle\bar{B}\gamma_\mu\gamma_5 B\rangle\langle u^\mu\rangle + \frac{\lambda}{2\tau}\langle\bar{B}\gamma_\mu\gamma_5 B\rangle\nabla^\mu\theta. \end{aligned} \tag{33}$$

This Lagrangian agrees with the one in Eq. (29) after expanding the functions W_i and taking only the lower orders into account. Higher powers of Q correspond to higher orders in the expansion of the W_i.

[4] In [5] a subset of the Lagrangian considered here has been discussed.

This time it is of particular interest what kind of modifications in the Lagrangian result if Q is eliminated via an equation of motion which includes the baryons. For simplicity we will restrict ourselves to the equation of motion which results from the Lagrangian

$$\delta\mathcal{L} = -Q_0 Q + \tau Q^2 + i\kappa Q\langle \bar{B}\gamma_5 B\rangle. \tag{34}$$

The pertinent equation of motion reads

$$-Q_0 + 2\tau Q + i\kappa\langle \bar{B}\gamma_5 B\rangle = 0 \tag{35}$$

with the solution

$$Q = \frac{1}{2\tau}\Big(Q_0 - i\kappa\langle \bar{B}\gamma_5 B\rangle\Big). \tag{36}$$

Reinserting the solution for Q into the Lagrangian gives rise to terms quartic in the baryons, e.g. $\langle \bar{B}\gamma_5 B\rangle\langle \bar{B}\gamma_5 B\rangle$. Since we restricted ourselves from the beginning to the one-baryon sector, we can drop these additional contributions. But in general such terms will renormalize the parameters of an effective theory with more baryons.

5 Summary

In this work we have shown the equivalence of two different frameworks which have been proposed to include the η' in chiral perturbation theory both in the purely mesonic sector and in the presence of the ground state baryon octet. In the first approach, one starts with an effective chiral Lagrangian which is invariant under axial $U(1)$ rotations. This is achieved by treating the vacuum angle θ as an external field $\theta(x)$ which transforms under $U(1)_A$ in such a way that it compensates the term added to the QCD Lagrangian by the anomaly.

In the second framework, one keeps the topological charge density Q as a background field within the effective theory. The chiral Lagrangian includes a term proportional to Q which is not invariant under $U(1)_A$ and reproduces the anomaly in the divergence of the axial-vector current. The field Q, on the other hand, is treated as $U(1)_A$ invariant and is eliminated via its classical equation of motion. The $U(1)_A$ invariance of Q is only fulfilled if one adds a parity violating piece θQ to the Lagrangian and proposes the same transformation law under $U(1)_A$ for θ as in the first scheme. The relation between the different sets of parameters in both Lagrangians is clarified for arbitrary high powers of Q. From our discussion it becomes clear, that the first procedure of starting with an $U(1)_A$ invariant Lagrangian is simpler, although the second framework might serve as a check for deriving the effective Lagrangian.

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